Interactions with local embeddability

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Basic notions

LEF (semi)group S

For every finite subset H of S there exists a finite (semi)group F_H and an injective map $f_H : H \to F_H$, such that for all $x, y \in H$ (with $xy \in H$) we have $(xy)f_H = (xf_H)(yf_H)$. The pair (F_H, f_H) is called *approximating* for H.

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Equivalent definitions

Let $\mathcal{F}$ be the class of finite semigroups.

- S is LEF;
- S is a model of $Th_\forall(\mathcal{F})$;
- S is embeddable into a model of $Th(\mathcal{F})$.

Example and non-example

Free semigroup $S_n = Sg\langle x_1, \dots, x_n \rangle$

Any finite subset H of S_n embeds into the semigroup

$$F_H = Sg\langle 0, x_1, \dots, x_n | \mathcal{R} \rangle$$

where $\mathcal{R}$ is the set of relations stating $0u = u0 = 0$ for all u and $w = 0$ for $l(w) \geq m$ where $m = \max\{l(h) | h \in H\}$.

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Semigroup $T = Sg\langle a, b | a^2 b = a \rangle$

The finite subset $H = \{a, b, ab, aba\}$ cannot embed into a finite semigroup. For a potential approximating pair (F, f) of H denote $c = af$ and $d = bf$. We would have $c^n = c^{n+r} \implies c^{n-1} = c^n d = c^{n+r} d = c^{n+r-1} \implies \ldots \implies c = c^{r+1} \implies cd = c^r \implies cdc = ccd = c$ which contradicts the fact that $aba \neq a$.

History and study of the concept

General and semigroup papers

- E. Gordon, A. Vershik, *Groups that are locally embeddable in the class of finite groups*, Algebra i Analiz **9**:1 (1997), 71–97;
- O. Belegradek, *Local embeddability*, Algebra and Discrete Mathematics **14**:1 (2012), 14–28;
- D. K., *Semigroups locally embeddable into the class of finite semigroups*, IJAC **33:5** (2023), 969–988;
- D. K., *Restrictions on local embeddability into finite semigroups*, Semigroup Forum **109** (2024), 148–166.

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- deeper analysis of LEF semigroup classes;
- direct generalisation of the LEF property, e.g. *soficity*;

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There are multiple possible further avenues of research, including:

- deeper analysis of LEF semigroup classes;
- direct generalisation of the LEF property, e.g. *soficity*;
- studying other possible properties inspired by definitions of LEF.

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A group is an LEF semigroup if and only if it is an LEF group.

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An inverse semigroup (i.e. a semigroup where for every x there exists a unique x^{-1} such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$) is an LEF semigroup if and only if it is locally embeddable into finite inverse semigroups.

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Proposition

A Clifford semigroup (i.e. inverse semigroup where $xx^{-1} = x^{-1}x$) is an LEF semigroup if and only if it is locally embeddable into finite Clifford semigroups.

...and failure to do so

Cancellativity

An example by A. Malcev, the cancellative semigroup $Sg\langle a, b, c, d, x, y, u, v | ax = by, cx = dy, au = bv\rangle$ is LEF but it is not embeddable into finite cancellative semigroups, a.k.a. groups.

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It follows from the fact that the relations would imply $cu = dv$ in a group.

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$\mathcal{J}$-triviality

The semigroup $Sg\langle a, b, c, e, x | xb = cx, ac = ca, ea = ae, ec = ce, aex = ax, xca = xe\rangle$ is LEF but it is not embeddable into finite $\mathcal{J}$-trivial semigroups.

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It follows from the fact that in finite $\mathcal{J}$-trivial semigroups we would have $z^r = z^{r+1}$ for any z and some power r, while we also have $xa^n x = xa^{n+1}xb$.

Soficity, p.1

Group case

For every finite subset H of the group G and $\epsilon \geq 0$ there exists a finite symmetric group Σ_n and a map $f : G \to \Sigma_n$, such that:

- $d((g_1 f)(g_2 f), (g_1 g_2) f) \leq \epsilon$ for all $g_1, g_2 \in H$;
- $d(g_1 f, g_2 f) \geq 1 - \epsilon$ for all distinct $g_1, g_2 \in H$;

where $d(x, y)$ is the Hamming metric on Σ_n.

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Semigroup case?

For every finite subset H of the semigroup S and $\epsilon \geq 0$ there exists a finite full transformation monoid T_n and a map $f : S \to T_n$, such that:

- $d((s_1 f)(s_2 f), (s_1 s_2) f) \leq \epsilon$ for all $s_1, s_2 \in H$;
- $d(s_1 f, s_2 f) \geq 1 - \epsilon$ for all distinct $s_1, s_2 \in H$;

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Soficity, p.2

Bicyclic monoid

The monoid $B = Mon\langle p, q | pq = 1\rangle$ fails to satisfy the definition of "semigroup soficity" due to the fact that $d(1, xy) = d(1, yx)$ in T_n.

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To see that, consider the set $K = \{k \in \{1, \dots, n\} | k(yx) = k\}$. We have $|K| = |Ky|$. Furthermore, for every $i \in Ky$ there exists $k \in K$ such that $i = ky$. Note that $i(xy) = k(yxy) = ky = i$. Thus, we have $d(1, yx) = 1 - \frac{|K|}{n} \geq d(1, xy)$. A dual argument gives us the desired equality.

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The monoid $B \cup \{id\}$ where id is an external identity satisfies the definition of "semigroup soficity".

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For more information see M. Kambites, *A large class of sofic monoids*, Semigroup Forum **91** (2015), 282–294.

A "converse" property

LWF (semi)group S

For every finite subset H of S there exists a finite (semi)group D_H and a map $d_H : D_H \to S$, such that $H \subseteq D_H d_H$ and for all $x', y' \in D_H$ with $x'd_H, y'd_H \in H$ it holds that $(x'y')d_H = (x'd_H)(y'd_H)$.

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Theorem (E. Gordon, A. Vershik, 1997)

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Theorem (E. Gordon, A. Vershik, 1997)

A group is LEF if and only if it is LWF.

Proposition

There is an LWF semigroup which is not LEF.

An LWF example...

...and more!

The semigroups $Sg\langle a, b, c, d, e, x | abx = acx, ac = cd, ae = ed, xcd = xe, aex = ax, xexb = bxex\rangle$ and $Sg\langle a, b, c, e, x | abx = acx, ac = ca, ae = ea, xcd = xe, aex = ax, xexb = bxex\rangle$ are not LEF. However, they are $\mathcal{J}$-trivial and LWF.

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The LWF property is not all-encompassing.

A usual suspect

The bicyclic monoid is not LWF.

Another "converse" property

Recall that LEF structures are exactly substructures of the models of $Th(\mathcal{F})$ where $\mathcal{F}$ is the class of finite semigroups. These can be alternatively seen as ultraproducts of finite semigroups.

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How can we characterise the structures which are *quotients* of ultraproducts of finite semigroups (UFS)?

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A non-example

The group $\mathbb{Z}$ is not isomorphic to a quotient of any UFS.

What is “finite” anyway?

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Free Burnside (semi)groups

- A group given by presentation $Gp\langle x_1, \ldots, x_m | R_r \rangle$ where $R_r = \{(w^r = 1) | w \in \{x_1^{\pm}, \ldots, x_m^{\pm}\}^*\}$;
- A semigroup given by presentation $Sg\langle x_1, \ldots, x_m | R_{r,n} \rangle$ where $R_{r,n} = \{(w^n = w^{r+n}) | w \in \{x_1, \ldots, x_m\}^*\}$.

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For large enough n, r both structures are infinite.

Proposition

There exists a periodic structure which is not LEF.

Other open problems and some progress on them

- More LEF interactions, like E-unitary inverse semigroups (free inverse semigroup is locally embeddable into finite E-unitary semigroups) and completely regular semigroups (completely simple semigroups are locally embeddable into finite completely simple semigroups);

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- Proving or disproving that inverse semigroups that are LWF are also LEF (Clifford semigroups and inverse semigroups with finite number of idempotents satisfy this property);
- Finding better definition of "semigroup soficity".

Thank you!